

Note on Centrally Extended $su(2/2)$ and Serre Relations

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Abstract

We point out that the nontrivial central extension of the superalgebra $su(2/2)$ is related to the some not so well-known Serre relations.

1 Introduction

Recently, the superalgebra $su(2/2)$ played an important role in the study of planar N=4 super Yang-Mills and in the construction of S-matrix for dynamic spin chains, cf. [1], and for latest references, [2]. In [1] was exploited the fact that the superalgebra $su(2/2)$ has an extraordinary feature, namely, it is the only basic classical Lie superalgebra¹ that has nontrivial central extension [4] (besides the trivial extension of $psu(n/n)$ to $su(n/n)$). This feature is related to the fact that the nontrivial central extension of the complexification $sl(2/2; \mathcal{C})$ may be obtained by contraction from the algebra $D(2, 1; \bar{\sigma})$.

The aim of this Note is to point out that the origin of the above phenomenon is in some not so well-known Serre relations for $sl(2/2; \mathcal{C})$. Actually, we pose the question more generally in order to stress how special the superalgebra $sl(2/2; \mathcal{C})$ is.

¹A basic classical Lie superalgebra is a Lie superalgebra which has a non-degenerate, even, supersymmetric, invariant bilinear form [3].

Serre relations [5] were introduced as an important tool for handling semi-simple Lie algebras. Using them one can reconstruct the full algebra starting from the Chevalley generators which correspond to the simple roots. One would expect that these relations would be used also for semi-simple Lie superalgebras. However, that was not the case. Initially, Lie superalgebras were given, constructed and classified in the full Cartan-Weyl basis using their realization in terms of finite-dimensional super-matrices [3]. Only in the development of q -superalgebras it became necessary to generalize Serre relations to q -deformations of Lie superalgebras in order to be able to use Drinfeld's theory of q -Serre relations [6]. Initially, it was thought that the generalization to the q -deformed case would be done just by superizing Drinfeld's q -Serre relations. However, in 1991, several sets of authors [7, 8, 9, 10] discovered independently that for some superalgebras it is necessary to introduce new Serre relations, which actually are necessary also in the undeformed case, $q = 1$.

The superalgebra $sl(2/2; \mathcal{C})$ is one of the superalgebras that need extra Serre relations, yet, it is unique in the sense that failing to impose them, one obtains a sensible result. Namely, one obtains the superalgebra $D(2, 1; \bar{\sigma})$.

This Note is organized as follows. In the next Section we present the defining relations for (q -deformed) superalgebras, including all Serre relations. Then we specialize to the cases of $sl(2/2; \mathcal{C})$ and $D(2, 1; \bar{\sigma})$ demonstrating their relationship.

2 Defining relations for (q -deformed) superalgebras

Let \mathcal{G} be a complex Lie superalgebra with a symmetrizable Cartan matrix $A = (a_{jk}) = A^d \hat{A}^s$, where $\hat{A}^s = (\hat{a}_{jk}^s)$ is a symmetric matrix, and $A^d = \text{diag}(\hat{d}_1, \dots, \hat{d}_\ell)$, $\hat{d}_k \neq 0$. Then the q - deformation $U_q(\mathcal{G})$ of the universal enveloping algebras $U(\mathcal{G})$ is defined [11] as the associative algebra over \mathcal{C} with generators X_j^\pm , H_j , $j \in J = \{1, \dots, \ell\}$ and with relations similar to the even case :

$$\begin{aligned} [H_i, H_j] &= 0, \quad [H_i, X_j^\pm] = \pm \hat{a}_{ij}^s X_j^\pm \\ [X_i^+, X_j^-] &= \delta_{ij} [H_i]_{q_i}, \quad q_i = q^{\hat{d}_i} \end{aligned} \tag{1}$$

([,] being the supercommutator),

$$(\text{Ad}_{q^\kappa} X_j^\pm)^{n_{jk}} (X_k^\pm) = 0, \text{ for } j \neq k, \quad \kappa = \pm; \quad (2)$$

and for every three simple roots, say, $\alpha_j, \alpha_{j\pm 1}$, such that $(\alpha_j, \alpha_j) = 0, (\alpha_{j\pm 1}, \alpha_{j\pm 1}) \neq 0, (\alpha_{j+1}, \alpha_{j-1}) = 0, (\alpha_j, \alpha_{j+1} + \alpha_{j-1}) = 0$, also holds:

$$[[X_j^\pm, X_{j-1}^\pm]_{q^\kappa}, [X_j^\pm, X_{j+1}^\pm]_{q^\kappa}] = 0 \quad (3)$$

where :

$$\begin{aligned} n_{jk} = & \begin{cases} 1 & \text{if } \hat{a}_{jj}^s = \hat{a}_{jk}^s = 0 \\ 2 & \text{if } \hat{a}_{jj}^s = 0, \hat{a}_{jk}^s \neq 0 \\ 1 - 2\hat{a}_{jk}^s/\hat{a}_{jj}^s & \text{if } \hat{a}_{jj}^s \neq 0 \end{cases} \end{aligned} \quad (4)$$

where in (1),(2) one uses the deformed supercommutator:

$$\begin{aligned} (\text{Ad}_{q^\kappa} X_j^\pm) (X_k^\pm) &= [X_j^\pm, X_k^\pm]_{q^\kappa} \equiv \\ &\equiv X_j^\pm X_k^\pm - (-1)^{p(X_j^\pm)p(X_k^\pm)} q^{\kappa(\alpha_j, \alpha_k)/2} X_k^\pm X_j^\pm \end{aligned} \quad (5)$$

The above is applicable to the even case, then relations (1),(2) for $\kappa = 1$ are the same as for $\kappa = -1$ and coincide with the usual even Serre relations. The necessity of the extra relations (3) (including the classical $q = 1$ case) was communicated to the author in May 1991 independently by Scheunert, Kac, and Leites. In the q -deformed case these relations were written for $U_q(sl(M/N))$ in [9] and [8], for $U_q(osp(M/2N))$ in [8], and in general in [10]. Here they are given as in [11], except that in [11] also the external supercommutator is given as deformed, while in fact it is not. The reason is that the would be deformation is given by the factor $q^{\kappa(\alpha_j + \alpha_{j-1}, \alpha_j + \alpha_{j+1})}$, and as can be easily checked the scalar product in the q -exponent is actually zero. (Later extra Serre relations were written also for the affine (q -deformed) Kac-Moody superalgebras [12, 13] and for any Lie superalgebra with Cartan matrix [14]. See also recent papers on the q -deformed affine superalgebra $D(2, 1; \bar{\sigma})$ [15] and the $su(2/2)$ Yangian [16] relevant in our current context and related to [1].)

Pictorially, one case of the situation with the three roots in (3) is given by the following (part of) Dynkin diagram:

$$\textcircled{1} --- \textcircled{*}_1 --- \textcircled{1} \quad (6)$$

where a circle represents an even root, while a crossed (gray) circle represents an odd root. The number at a node gives the coefficient with which the corresponding simple root enters the decomposition of the highest root.

3 The cases of $sl(2/2; \mathcal{C})$ and $D(2, 1; \bar{\sigma})$

The three roots in (3) and three nodes in (6) by themselves determine the root system and Dynkin diagram of the superalgebra $sl(2/2; \mathcal{C})$. We recall that this superalgebra is 15-dimensional (over \mathcal{C}), the even subalgebra being seven-dimensional:

$$sl(2/2; \mathcal{C})_{\bar{0}} \cong sl(2, \mathcal{C}) \oplus sl(2, \mathcal{C}) \oplus \mathcal{C}$$

while the odd part $sl(2/2; \mathcal{C})_{\bar{1}}$ is eight-dimensional. We choose a distinguished root system (with one odd simple root) corresponding to (6), the simple roots being denoted as $\alpha_1, \alpha_2, \alpha_3$, of which α_2 is odd, the other - even. The positive roots are:

$$\begin{aligned} \Delta^+ &= \Delta_{\bar{0}}^+ \cup \Delta_{\bar{1}}^+, & \Delta_{\bar{0}}^+ &= \{\alpha_1, \alpha_3\}, \\ & & \Delta_{\bar{1}}^+ &= \{\alpha_2, \alpha_1 + \alpha_2, \alpha_2 + \alpha_3, \alpha_1 + \alpha_2 + \alpha_3\} \end{aligned} \quad (7)$$

(Note that $\alpha_1 + \alpha_2 + \alpha_3$ is the highest root.)

We denote the Chevalley generators as X_j^\pm, H_j , $j = 1, 2, 3$. Then the Cartan subalgebra of $sl(2/2; \mathcal{C})$ is spanned by H_j , $j = 1, 2, 3$, though often instead of H_2 , (related to the odd root α_2), is used the central generator $K = H_1 + 2H_2 - H_3$. (In a matrix realization of 4×4 matrices, $K \sim I_4$, the unit 4×4 matrix.)

The defining relations (1) and (2) (for $q = 1$) are clear. We are interested in the new relations (3) which we write out explicitly for $q = 1$:

$$[[X_2^\pm, X_1^\pm], [X_2^\pm, X_3^\pm]]_+ = 0. \quad (8)$$

We pass now to the superalgebra $D(2, 1; \bar{\sigma})$. We recall that this superalgebra is 17-dimensional, the even subalgebra being nine-dimensional:

$$sl(2/2; \mathcal{C})_{\bar{0}} = sl(2, \mathcal{C}) \oplus sl(2, \mathcal{C}) \oplus sl(2, \mathcal{C})$$

while the odd part $sl(2/2; \mathcal{C})_{\bar{1}}$ is eight-dimensional [3]. We choose a distinguished root system, the simple roots being denoted as $\alpha_1, \alpha_2, \alpha_3$, of which α_2 is odd, the other - even. The positive roots are:

$$\begin{aligned} \Delta^+ &= \Delta_{\bar{0}}^+ \cup \Delta_{\bar{1}}^+, & \Delta_{\bar{0}}^+ &= \{\alpha_1, \alpha_3, \alpha_1 + 2\alpha_2 + \alpha_3\}, \\ & & \Delta_{\bar{1}}^+ &= \{\alpha_2, \alpha_1 + \alpha_2, \alpha_2 + \alpha_3, \alpha_1 + \alpha_2 + \alpha_3\} \end{aligned} \quad (9)$$

We note that $\tilde{\alpha} = \alpha_1 + 2\alpha_2 + \alpha_3$ is the highest root, and the root system corresponds to the following Dynkin diagram:

$$\begin{array}{c} \circ \\ 1 \end{array} \leftarrow \bigotimes_2 \begin{array}{c} \circ \\ 1 \end{array} \rightarrow \begin{array}{c} \circ \\ 1 \end{array} \quad (10)$$

For this superalgebra the defining relations are only (1) and (2) for $q = 1$, since the Dynkin diagram is not of the type (6). Our interest is to give an explicit expression for the Cartan-Weyl generators $X_{\tilde{\alpha}}^{\pm}$ corresponding to the highest root $\tilde{\alpha}$. After a simple calculation we obtain:

$$X_{\tilde{\alpha}}^{\pm} = [[X_2^{\pm}, X_1^{\pm}], [X_2^{\pm}, X_3^{\pm}]]_+ \quad (11)$$

Identifying the simple roots in (7) and (9) and comparing (8) and (11) we see that the superalgebras $sl(2/2; \mathcal{C})$, $D(2, 1; \bar{\sigma})$ differ by the extra Serre relations (8) which are needed for $sl(2/2; \mathcal{C})$ but are absent for $D(2, 1; \bar{\sigma})$. Consequently, we can pass from $D(2, 1; \bar{\sigma})$ to $sl(2/2; \mathcal{C})$ by a contracting procedure in which the root vectors $X_{\tilde{\alpha}}^{\pm}$ (11) are sent to zero. However, we may use a more general contraction procedure, as in [1], in which the root vectors are replaced by central elements, (not necessarily zero), i.e., setting:

$$X_{\tilde{\alpha}}^{\pm} \longrightarrow \mathcal{C}^{\pm} \in \mathcal{C} \quad (12)$$

we obtain the central extension of $sl(2/2; \mathcal{C})$ found in [4].

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